

Making Recommendations Bandwidth Aware

Linqi Song and Christina Fragouli

Abstract

This paper asks how much we can gain in terms of bandwidth and user satisfaction, if recommender systems became bandwidth aware and took into account not only the user preferences, but also the fact that they may need to serve these users under bandwidth constraints, as is the case over wireless networks. We formulate this as a new problem in the context of index coding: we relax the index coding requirements to capture scenarios where each client has preferences associated with messages. The client is satisfied to receive any message she does not already have, with a satisfaction proportional to her preference for that message. We consistently find, over a number of scenarios we sample, that although the optimization problems are in general NP-hard, significant bandwidth savings are possible even when restricted to polynomial time algorithms.

Index Terms

Pliable index coding, recommendation, bandwidth constraint.

I. INTRODUCTION

Recommender systems decide which content to offer to users so as to maximize a benefit (for instance, in advertising networks the benefit could be the profit gained from the ad placement) [1], [2]. These recommendations are currently oblivious to the cost of distributing the content from the server to the points of consumption, which however forms many times the point of failure: unsatisfactory delivery is identified as a core threat to the user experience and has already caused loss of billions of revenue dollars [3]. Wireless consumption in particular, that is increasingly gaining momentum, is inherently subject to bandwidth constraints.

In this paper, we ask: how much could we gain in terms of bandwidth and user satisfaction, if recommendation systems became bandwidth aware, and took into account not only the user preferences, but also the fact that they need to serve these users under bandwidth constraints?

L. Song and C. Fragouli are with the Department of Electrical Engineering, University of California, Los Angeles (UCLA). Email: {songlinqi, christina.fragouli}@ucla.edu. Part of this paper was submitted to 2017 IEEE International Symposium on Information Theory (ISIT 2017).

We formulate this as a new problem in the context of index coding. The index coding problem [4], [5], [6] considers a server with m messages and n clients. Each client has as side-information a subset of the messages and requires a specific message she does not have. The server can make error-free broadcast transmissions to all clients; the goal is to minimize the number of transmissions so that all clients are satisfied.

We relax the index coding requirements to capture scenarios where each client has preferences associated with messages: a client can now be satisfied by receiving any message she does not already have; however, the benefit we get is proportional to how high her preference, for the message she gets, is. For instance, consider wireless stations serving sale coupons inside a shopping mall: a client walking outside a shop would be happy to receive a coupon she does not already have, but would be happier to receive (and more likely to use) a coupon closer to her interests. We note that the side-information setup fits well with the recommender systems framework [2]: collecting side information about the clients and keeping track of previous content served is an integral part of recommender systems; it is a natural step to leverage this side information, not only to inform recommendations, but to also increase the communication efficiency so as to extract more benefits under communication constraints. But for the amount of interesting work in index coding (eg., [4], [5], [6], [7], [8]), this is the first paper as far as we know that explores trade-offs between user satisfaction and bandwidth.

A challenge we faced when setting as our goal to evaluate potential benefits, is that these depend on the preferences model we use. There exist numerous models for expressing preferences and for taking decisions based on them; clearly we cannot exhaustively investigate all possible ranking models. We opted to sample a few models that we thought were representative, with the hope of finding consistent trends across them. One model we investigated uses the Borda count, that has each client sort m messages according to her preferences, and assigns to a message ranked i by a client a score of $m + 1 - i$ [9]. We also considered a bimodal preferences model, where a fraction of the messages are much more preferable than the remaining. More generally, we considered an arbitrary score model, where each message gets an arbitrary score w_{ij} by a client.

To calculate the aggregate benefit, we count only the highest-preference message we have served to each client. This is motivated from that, if a client at a certain time can see only one video (or read one article or click one ad), although her device may have downloaded multiple items, she will only see her most preferred one, and we will collect the corresponding benefit.

This benefit model aligns well with the index-coding rationale, where only the one message the client wants counts.

The main contribution of this paper is to examine the trade-off between benefit and bandwidth across three scenarios, both theoretically and numerically using designed algorithms. We first provide results for the case where there is no side information for each client and each client has a full ranking of the messages. We show that the problem is NP-hard, however a simple greedy polynomial time algorithm can achieve an approximation ratio 1.58. Moreover, we provide upper and lower bounds of optimal performance as well as an average case analysis, both indicating diminishing returns: the benefits increase with the number t of transmissions only by a multiplicative factor $1 - 1/t$.

The second scenario investigates the case where each client has side-information of the same cardinality and a partial ranking over the desired messages. We prove lower bounds on the optimal benefits, and design a polynomial-time algorithm that achieves a $O(1)$ approximation ratio.

The third scenario is to consider a general case where each client has arbitrary size side information. This problem is hard to approximate within a ratio of $n^{1-\epsilon}$ for any $\epsilon > 0$. For this case we establish a connection with the maximum weighted independent set problem; we design and evaluate a heuristic coded algorithm that leverages this connection.

We evaluate our algorithms numerically over synthetic and real world data sets (Yahoo! advertiser bidding data sets [10]). We find that even with one transmission we can in many cases already achieve half of the maximum benefit possible; and in general, we can achieve 80% of the benefit with less than 10% of the transmissions we would need to achieve 100% of it. We also find that leveraging side information to make coded transmissions, can in some cases enable to double the benefit over uncoded transmissions.

The paper is organized as follows. We formulate the bandwidth aware recommendation problem in Section II, by analytically deriving polynomial time algorithms and performance bounds in Sections III-V. We present the numerical experiments in Section VI and conclude the paper in Section VII.

II. SETUP AND PROBLEM FORMULATIONS

A. Setup

We assume that a server has m messages b_1, b_2, \dots, b_m and n clients c_1, c_2, \dots, c_n . We will sometimes say message j instead of b_j , and similarly, client i instead of c_i . We use the notation $[m]$ to denote a set $\{1, 2, \dots, m\}$. Each client $i \in [n]$ already knows (has as side information) a subset of the m messages; we denote by $S_i \subseteq [m]$ the side information of client i (S_i could be the empty set), and by $R_i = [m] \setminus S_i$ the messages that the client may request (does not have).

Broadcast transmissions and coding: The server is connected to the clients through error-free broadcast transmissions; that is, all clients perfectly receive each server transmission. We assume that each message b_j , $1 \leq j \leq m$, takes values in a finite field \mathbb{F}_q . During the l -th transmission, the server transmits $x_l = \sum_{j \in [m]} a_{lj} b_j$, where $a_{lj} \in \mathbb{F}_q$ are constant coefficients and the addition/multiplication operations are performed in \mathbb{F}_q . Thus the server transmits either one of the uncoded messages b_j , or a linear combination of some of the messages. Assume the server broadcasts t transmissions $X = \{x_1, x_2, \dots, x_t\}$, we will denote by $D_i = \phi_i(S_i, X)$ the set of new messages that client i can decode, where $\phi_i(S_i, X)$ is the decoding function.

Scores and benefit B: Each client i has a *rank or preference* $\pi_i(j)$ for each of the messages j in her request set R_i ; accordingly, we get a *message score* $s_i(j)$ when message j is decoded by client i . Sometimes we omit the ranking and assume that the message scores are given directly. A client i has *client score* $s(i) = \max_{j \in D_i} s_i(j)$; that is, we only count the message of highest score among the D_i messages she decodes. If D_i is empty we set $s(i) = 0$. The *benefit B* we get is the aggregate client score $B = \sum_{i \in [n]} s(i)$. We considered the following models for scores:

- The *Borda count* method assumes that the ranking is a permutation of the set R_i and calculates message scores as $s_i(j) = |R_i| + 1 - \pi_i(j)$ (a message ranked first gives score $|R_i|$, ranked second gives score $|R_i| - 1$, etc.)
- The *bimodal score* assumes that a fraction F of the messages are much more desirable than the remaining $(1 - F)$ fraction. In particular we assume that the ranking $\pi_i(j)$ is a permutation of the $|R_i|$ messages, and we set $s_i(j) = G(m + 1 - \pi_i(j))$ if $\pi_i(j) \leq F|R_i|$ and $s_i(j) = m + 1 - \pi_i(j)$ otherwise. The parameter G determines how separated (bimodal) the two sets of messages are.
- The general model assigns an *arbitrary weight* to each score $s_i(j) = w_{ij}$.

Performance metrics: We are interested in the tradeoff between the number t of broadcast transmissions and the corresponding achievable benefit B .

B. Problem Formulations

We here first express in a unified notation the index and pliable index coding problem that have been examined in the literature before, and then introduce the new formulations we will examine through theoretical analysis in this paper.

Past Formulations

Index Coding: Each client requests a specific message that she does not have; if client i would like to receive b_{j_i} , we set $s_i(j) = 1$ for $j = j_i$ and zero otherwise ($\pi_i(j) = |R_i|$ for $j = j_i$ and $\pi_i(j) = |R_i| + 1$ otherwise). Thus $s(i)$ takes values either 0 or 1, depending on whether client i can decode b_{j_i} or not, and $0 \leq B \leq n$. Index coding asks for the minimum number of transmissions to achieve the maximum benefit $B = n$ possible, i.e., so that all clients receive the message they have requested.

This problem is NP hard and requires in the worst case $\Omega(n)$ transmissions [5], [11], [6], and almost surely $\Theta(\frac{n}{\log(n)})$ transmissions for random graphs [12], [13].

Pliable Index Coding: Each client is happy to receive any message she does not have (without any preference). We thus set $s_i(j) = 1$, for all i and $j \in R_i$ ($\pi_i(j) = |R_i|$, for all i and $j \in R_i$), $s(i)$ takes value 1 if client i decodes any one message in R_i , and $0 \leq B \leq n$. Pliable index coding asks for the minimum number of transmissions to achieve benefit $B = n$.

This problem is NP hard, but there exist polynomial time algorithms that require in the worst case $O(\log^2 n)$ transmissions [14], [15].

New Formulations

The following formulations describe some scenarios for which we derive theoretical results. In each case, we ask what is the benefit B with t transmissions.

P1. No side information and full ranking: No side information implies that $R_i = [m]$. We consider the Borda count, where $\pi_i(j)$ defines for each client i a permutation of $[m]$, and $s_i(j) = m + 1 - \pi_i(j)$ is also a permutation of $[m]$. Thus $0 \leq B \leq nm$.

P2. Equal size side information and partial ranking: We assume that $|S_i| = m - k$ for all clients, $\pi_i(j)$ defines for client i a permutation over the remaining k messages, and $s_i(j) = k + 1 - \pi_i(j)$. In this case $0 \leq B \leq nk$.

P3. Arbitrary size side information and score: If the size of the side information set for each receiver is arbitrary, we cannot use a permutation of R_i as ranking of the messages to

Algorithm 1 Greedy algorithm for P1.

- 1: **Input:** ranking matrix Π and number of columns to select t .
 - 2: **Output:** a set of columns \mathcal{T} .
 - 3: **Initialization:** set $\mathcal{T} = \emptyset$, $B_{\mathcal{T}} = 0$.
 - 4: **for** $\tau = 1 : t$ **do**
 - 5: $j = \arg \max_{j' \in [m] \setminus \mathcal{T}} B_{\mathcal{T} \cup \{j'\}}$
 find a column j to maximize the benefit given current $\tau-1$ selected columns \mathcal{T} .
 - 6: $\mathcal{T} = \mathcal{T} \cup j$.
 - 7: **end for**
-

calculate the score, as it would give unfair weight to the different clients. We assume instead that (fair) scores $s_i(j) = w_{ij}$ are provided as input.

III. NO SIDE INFORMATION (P1)

This is the simplest case we examine. This problem is close to the rank aggregation problems studied in the literature [9], [16], [17], [18], the difference being that only the highest ranked message a client receives counts towards the total benefit. Interestingly, while the rank aggregation problem is polynomial time using the Borda count optimal rule, taking into account only the highest score message makes the problem NP-hard. We next describe our results, and provide the proofs in the Appendix A.

The problem is NP-hard.

Theorem 1. *The full ranking with the Borda score model and no side information problem (P1) is NP-hard.*

The proof uses a reduction from the set cover problem (see Appendix A-A).

Greedy selection achieves an approximation ratio of 1.58. We collect the preferences into an $n \times m$ ranking matrix Π , where the (i, j) entry is the rank of message j by client i , i.e., $\pi_i(j)$. That is, each row of this matrix expresses the ranking of messages by a client. For $t = 1$, we simply need to find the column of the matrix Π , whose elements have the highest sum (this would be the benefit B). For $t > 1$, we need to select t columns in a set \mathcal{T} such that $B_{\mathcal{T}}$ is as large as possible (we denote by $B_{\mathcal{T}}$ the benefit from a choice of a set of columns \mathcal{T}). Alg. 1 describes a straightforward greedy algorithm, that is sufficient to achieve a constant approximation ratio. Let \mathcal{T}^* be the optimal selection of t columns, and $B^* \triangleq B_{\mathcal{T}^*}$ the optimal benefit achieved by this selection over this problem instance. We have the following theorem.

Theorem 2. *For any P1 problem instance with no side information and full ranking, given that we can make t broadcast transmissions, Alg. 1 can achieve an approximation ratio at least $1/(1 - (1 - \frac{1}{t})^t)$, namely,*

$$\frac{B_{\mathcal{T}}}{B^*} \geq 1 - (1 - \frac{1}{t})^t. \quad (1)$$

The proof is given in Appendix A-B. From Theorem 2, we can see that for any t , the approximation ratio is bounded by a constant factor $1/(1 - \frac{1}{e}) = 1.58$.

Bounds on the optimal benefit B^* .

Theorem 3. *Consider the full ranking (Borda score) with no side information problem P1 with m messages, n clients and t transmissions. For all instances of P1*

$$\frac{tn(m+1)}{t+1} \leq B^*. \quad (2)$$

Moreover, if $n \geq 6t \log(m)$, there exist instances such that

$$B^* \leq (1 + \delta) \frac{tn(m+1)}{t+1}, \quad (3)$$

where $\delta = \sqrt{\frac{6t \log(m)}{n}}$.

The proof is in Appendix A-C. We underline that the upper bound does not apply for all instances (it is trivial to create instances where we get $B^* = nm$) but only for the worst case instances. Note that if δ is small the lower and upper bound have the same order of magnitude, which implies that the bounds become tight. Moreover, if the number of clients increases to $n > O(t^3 \log(m))$, then the (worst case instances) upper bound can be simplified to $O(mn(1 - \frac{1}{t}))$, which is close to the optimal nm . In particular, if we consider all possible $n = m!$ clients that have distinct rankings (permutations of $[m]$), then the benefit achieved by any t -selection is $\frac{tn(m+1)}{t+1}$.

Average case analysis. We here assume uniform distribution over the client rankings, and calculate the expected benefits. In particular, the ranking matrix Π is generated as follows: for each row (client ranking), select uniformly and independently a permutation from all $m!$ permutations of $[m]$ (with repetition).

Theorem 4. *The average benefit (averaged over the ranking matrices Π) in P1 using the Borda*

score model satisfies:

$$\mathbb{E}_{\Pi} B \geq \mu + n\Delta(m, n, t),$$

where $\mu = \frac{tn(m+1)}{t+1}$, $\Delta(2, n, 1) = \frac{1}{\sqrt{2\pi n}}$ for even n , $\Delta(2, n, 1) = \frac{1}{\sqrt{2\pi(n-1)}}$ for odd $n \geq 3$, $\Delta(m, n, t) \approx \frac{1}{\sqrt{2\pi n}}\sigma(m, t)$ for large n , and $\sigma(m, t) = \sqrt{\frac{(m+1)(m-t)t}{(t+1)^2(t+2)}}$ is the standard deviation of a client's score distribution when we randomly select t columns.

Note that we have already shown in Theorem 3 that the worst case benefit is μ ; the above theorem shows that the average is higher by at least $n\Delta(m, n, t)$. The proof is provided in Appendix A-D.

Illustrating example. Consider an instance with $n = 5$ clients, $m = 4$ messages and the 5×4 ranking matrix

$$\Pi = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & 1 & 2 & 3 \\ 4 & 2 & 1 & 3 \\ 4 & 2 & 3 & 1 \\ 4 & 1 & 2 & 3 \end{bmatrix}. \quad (4)$$

We can see that, to serve to all clients their first preference (benefit $B=20$) we need $t = 4$ transmissions, yet with only $t = 1$ transmission (second column) we can already serve to all clients either their first or their second preference (benefit $B = 17$). Moreover, in a situation where the server recommends to send to each client their first preference but only one of these messages is delivered (in time) because of bandwidth constraints, we would in the worst case achieve benefit $B = 8$ (e.g., only the first message is delivered); thus taking into account the bandwidth constraints can more than double the benefit. This difference can be magnified proportionally to a parameter G , if instead of Borda count we used bimodal score model with gain factor G .

IV. EQUAL SIZE SIDE INFORMATION (P2)

We now look at the case where all clients have side information of the same size $|S_i| = m - k$ and thus $|R_i| = k$, $\forall i$. We again assume Borda count scores.

Bounds on the optimal benefit: We are interested in the optimal benefit B^* we can achieve with t transmissions (recall that $0 \leq B \leq kn$). We next prove a lower bound on the performance of the optimal algorithm through dynamic programming.

Theorem 5. *The optimal benefit B^* satisfies*

$$B^* \geq \begin{cases} \frac{kn}{4e} & \text{for } t = 1, \\ \frac{kn}{e} \left(\frac{1}{2} - \frac{1}{8e} + \frac{1}{16e^2} \right) & \text{for } t = 2, \\ \frac{kn}{e} \left(\frac{3}{4} - \frac{1}{4e} + \frac{1}{16e^2} \right) & \text{for } t = 3, \\ \frac{kn}{e} \left(1 - \frac{5}{8e} + \frac{13}{64e^2} \right) & \text{for } t = 4, \\ kn \left(1 - \frac{4e}{t} + \frac{12e}{t^2} \right) & \text{for } 5 \leq t < k \\ kn & \text{for } t = k. \end{cases} \quad (5)$$

The proof of this theorem is constructive: we provide a randomized algorithm and show that it achieves on average the performance prescribed in the theorem, which implies that the optimal performance can only be better. The approximation ratio of this scheme is $O(1)$, as the best achievable benefit is kn . We also note that if $t = k$, we can easily achieve $B = kn$: the server can use an MDS erasure correcting code to create k linear combinations to transmit, so that each client using her side information can solve for the k messages she misses. We next show the proof of the theorem.

- For $t = 1$, assume that the server makes the transmission $x_1 = a_1b_1 + a_2b_2 + \dots + a_mb_m$ where $a_j, j \in [m]$, are the constant coding coefficients (we will call the vector $\mathbf{a} = (a_1, a_2, \dots, a_m)$ the coding vector). Assume we select iid random values for the coding coefficients, setting $a_j = 1$ with probability $1/k$, and $a_j = 0$ otherwise.

Claim 1: There exists a binary coding vector \mathbf{a}^ξ that enables at least $\frac{n\xi}{ke}$ clients to decode a message in their request set they have ranked less than or equal to ξ , for $\xi = 1, 2, \dots, k$, and thus to achieve a benefit of at least $\frac{(k+1-\xi)n\xi}{ke}$.

Proof. Without loss of generality, assume that client i has the request set $R_i = \{b_{l_1}, b_{l_2}, \dots, b_{l_k}\}$ with ranking $\pi(l_1) = 1, \pi(l_2) = 2, \dots, \pi(l_k) = k$. Client i can decode a message with rank at least ξ , i.e., can decode some b_{l_j} with $j \leq \xi$, if and only if $a_{l_j} = 1$ and $a_{l_1} = a_{l_2} = \dots = a_{l_{j-1}} = a_{l_{j+1}} = \dots = a_{l_k} = 0$. Indeed, we can then express the server transmission as

$x_1 = b_{l_j} + \sum_{l \in S_i} a_l b_l$; client i can remove from x_1 the part $\sum_{l \in S_i} a_l b_l$ using her side information, and decode b_{l_j} . The probability that such an event happens is:

$$\binom{\xi}{1} \frac{1}{k} \left(1 - \frac{1}{k}\right)^{k-1} \leq \frac{\xi}{ke} \triangleq p_\xi. \quad (6)$$

Hence, randomly selecting a coding vector would enable on average $np_\xi = \frac{n\xi}{ke}$ clients to decode messages of rank no more than ξ , and thus, from the averaging principle, there exists at least one coding vector \mathbf{a}^ξ that also enables this. \square

• For $t > 1$, we consider the following dynamic programming problem. We consider t stages, each corresponding to one transmission. At stage τ , $1 \leq \tau \leq t$, the server can select one of k actions, which of the k possible \mathbf{a}^ξ vectors to use. In particular, we proceed as follows:

– At the beginning of stage 1, no transmission has yet been made and there are $n_1 = n$ clients in the system. The server chooses an action $\xi_1 \in [k]$, i.e., uses the coding vector \mathbf{a}^{ξ_1} to make a transmission. From Claim 1, this transmission enables $\frac{n_1 \xi_1}{ke}$ clients to decode a message that they have ranked less than or equal to ξ_1 , and thus, we can achieve a benefit $B_1 \geq \frac{(k+1-\xi_1)n\xi_1}{ke}$. We remove these $\frac{n_1 \xi_1}{ke}$ clients from the system and denote the remaining number of clients by $n_2 = n_1(1 - \frac{\xi_1}{ke})$.

– At the beginning of stage 2, we only consider the n_2 clients; similarly to before, the server chooses an action $\xi_2 \in [k]$ to enable $\frac{n_2 \xi_2}{ke}$ clients decode a message ranked less than or equal to ξ_2 . At this point we have achieved benefit $B_2 \geq B_1 + \frac{(k+1-\xi_2)n_2 \xi_2}{ke}$. We remove these $\frac{n_2 \xi_2}{ke}$ clients from the system and denote the remaining number of clients by $n_3 = n_2(1 - \frac{\xi_2}{ke})$.

– Continuing along the same lines, at the beginning of stage $\tau = 3, 4, \dots, t$, we have n_τ clients to consider; the server chooses an action ξ_τ that enables to achieve benefit $B_\tau \geq B_{\tau-1} + \frac{(k+1-\xi_\tau)n_\tau \xi_\tau}{ke}$; and we set $n_{\tau+1} = n_\tau(1 - \frac{\xi_\tau}{ke})$.

Let $J_\tau(n_\tau)$ be the benefit the n_τ clients can receive for the *remaining* $t + 1 - \tau$ stages. We have the following Bellman equation:

$$J_\tau(n_\tau) = \max_{\xi_\tau \in [k]} \left\{ (k+1-\xi_\tau) \frac{\xi_\tau n_\tau}{ke} + J_{\tau+1}(n_\tau(1 - \frac{\xi_\tau}{ke})) \right\}, \quad (7)$$

with $J_{t+1}(n_{t+1}) = 0$ (as we will only make t transmissions).

From the above equation, we can see that the benefit achieved using this scheme is $B = J_1(n)$.

- If $t = 1$, we set $\xi_1 = \frac{k}{2}$ (k even) and $\xi_1 = \frac{k+1}{2}$ (k odd), and get $J_1(n) \geq \frac{kn}{4e}$.
- If $t = 2$, we set $\xi_1 = \lceil \frac{k}{2}(1 - \frac{1}{4e}) \rceil$ and $\xi_2 = \lceil \frac{k}{2} \rceil$, and get $J_1(n) \geq \frac{kn}{e}(\frac{1}{2} - \frac{1}{8e} + \frac{1}{16e^2})$.

- If $t = 3$, we set $\xi_1 = \lceil \frac{k}{2}(1 - \frac{1}{2e}) \rceil$, $\xi_2 = \lceil \frac{k}{2}(1 - \frac{1}{4e}) \rceil$ and $\xi_3 = \lceil \frac{k}{2} \rceil$, and get $J_1(n) \geq \frac{kn}{e}(\frac{3}{4} - \frac{1}{4e} + \frac{1}{16e^2})$.

- If $t = 4$, we set $\xi_1 = \lceil \frac{k}{2}(1 - \frac{3}{4e}) \rceil$, $\xi_2 = \lceil \frac{k}{2}(1 - \frac{1}{2e}) \rceil$, $\xi_3 = \lceil \frac{k}{2}(1 - \frac{1}{4e}) \rceil$ and $\xi_4 = \lceil \frac{k}{2} \rceil$, and get $J_1(n) \geq \frac{kn}{e}(1 - \frac{5}{8e} + \frac{13}{64e^2})$.

- For $t > 4$, we have the following claim.

Claim 2: $J_\tau(n) \geq kn(1 - \frac{4e}{t+1-\tau} + \frac{12e}{(t+1-\tau)^2})$ for $t > 4$.

Proof. We use the backward induction method for the last 5 stages. By setting $\xi_{t-4} = \lceil \frac{k}{2}(1 - \frac{1}{e} + \frac{5}{8e^2}) \rceil$, $\xi_{t-3} = \lceil \frac{k}{2}(1 - \frac{3}{4e}) \rceil$, $\xi_{t-2} = \lceil \frac{k}{2}(1 - \frac{1}{2e}) \rceil$, $\xi_{t-1} = \lceil \frac{k}{2}(1 - \frac{1}{4e}) \rceil$ and $\xi_t = \lceil \frac{k}{2} \rceil$, we can get $J_{t-4}(n_{t-4}) \geq \frac{kn_{t-4}}{e}(\frac{5}{4} - \frac{9}{8e} + \frac{39}{64e^2}) \doteq 0.33kn_{t-4}$. Therefore, we have the initial condition:

$$J_{t-4}(n_{t-4}) \doteq 0.33kn_{t-4} \geq kn_{t-4}(1 - \frac{4e}{t+1-(t-4)} + \frac{12e}{(t+1-(t-4))^2}) \doteq 0.13kn_{t-4}.$$

Assume that $J_{\tau+1}(n_{\tau+1}) \geq kn_{\tau+1}(1 - \frac{4e}{t+1-(\tau+1)} + \frac{12e}{(t+1-(\tau+1))^2})$ holds for $\tau + 1$, then consider $J_\tau(n_\tau)$ ($\tau < t - 4$):

$$\begin{aligned} J_\tau(n_\tau) &= \max_{\xi_\tau \in [k]} \{ (k+1 - \xi_\tau) \frac{\xi_\tau n_\tau}{ke} + J_{\tau+1}(n_\tau(1 - \frac{\xi_\tau}{ke})) \} \\ &\geq \max_{\xi_\tau \in [k]} \{ (k+1 - \xi_\tau) \frac{\xi_\tau n_\tau}{ke} + kn_\tau(1 - \frac{\xi_\tau}{ke})(1 - \frac{4e}{t+1-(\tau+1)} + \frac{12e}{(t+1-(\tau+1))^2}) \} \\ &\geq kn_\tau(1 - \frac{4e}{t+1-\tau} + \frac{12e}{(t+1-\tau)^2}), \end{aligned}$$

where the first inequality holds due to the hypothesis and the property of the Bellman equation; the second inequality holds by setting ξ_τ to be an integer between $\xi' = \frac{2ke}{t-\tau} - \frac{6ke}{(t-\tau)^2} + 1/2$ and $\xi'' = \frac{2ke}{t-\tau} - \frac{6ke}{(t-\tau)^2} - 1/2$. If we define $f(\xi) = (k+1 - \xi) \frac{\xi n_\tau}{ke} + kn_\tau(1 - \frac{\xi}{ke})(1 - \frac{4e}{t+1-(\tau+1)} + \frac{12e}{(t+1-(\tau+1))^2})$, then we have $f(\xi_\tau) \geq \min\{f(\xi'), f(\xi'')\} \geq kn_\tau(1 - \frac{4e}{t+1-\tau} + \frac{12e}{(t+1-\tau)^2})$. Therefore, claim 2 holds. By setting $\tau = 1$, we get $J_1(n) \geq kn(1 - \frac{4e}{t} + \frac{12e}{t^2})$. \square

Algorithm 2: We base our proposed algorithm in this case, as shown in Alg. 2, on the randomized algorithm described in the proof of Theorem 5 that operates in rounds, and in each round selects what coding vector to transmit so as to satisfy a certain fraction of clients. The only random step in this algorithm is the selection of a binary coding vector in Claim 1; however, we can easily derandomize it using a deterministic algorithm in polynomial time: we sequentially visit the entries of the coding vector and decide whether to assign value 0 or 1 depending on how the benefit would increase, as described in detail Appendix B.

Benefits of Coding: As is the case in index coding, leveraging side information enables to use coding and convey through the same transmission different messages to clients. We next

Algorithm 2 Dynamic programming algorithm for solving P2.

```

1: Input: number of messages  $m$ , number of clients  $n$ , request sets  $R_i, \forall i \in [n]$ , ranking  $\pi_i(j), \forall i \in [n], j \in R_i$ ,
   size of request set  $k$ , and number of selections  $t$ .
2: Output: coding matrix  $\mathbf{A} \in \{0, 1\}^{t \times m}$ .
3: Initialization: set the client set  $\mathcal{N} = [n]$ ;
4: for  $\tau = 1 : t$  do
5:   if  $\tau = t$  then
6:     Set ranking threshold  $\xi_\tau = \lceil \frac{k}{2} \rceil$ .
7:   else if  $\tau = t - 1$  then
8:     Set ranking threshold  $\xi_\tau = \lceil \frac{k}{2}(1 - \frac{1}{4e}) \rceil$ .
9:   else if  $\tau = t - 2$  then
10:    Set ranking threshold  $\xi_\tau = \lceil \frac{k}{2}(1 - \frac{1}{2e}) \rceil$ .
11:   else if  $\tau = t - 3$  then
12:    Set ranking threshold  $\xi_\tau = \lceil \frac{k}{2}(1 - \frac{3}{4e}) \rceil$ .
13:   else if  $\tau = t - 4$  then
14:    Set ranking threshold  $\xi_\tau = \lceil \frac{k}{2}(1 - \frac{1}{e} + \frac{5}{8e^2}) \rceil$ .
15:   else
16:    Set ranking threshold  $\xi_\tau = \lceil \frac{2ke}{t-\tau} - \frac{6ke}{(t-\tau)^2} - \frac{1}{2} \rceil$ .
17:   end if
18:   Find a row coding vector  $\mathbf{a}_\tau$  as the  $\tau$ -th row of  $\mathbf{A}$  with respect to the ranking threshold  $\xi_\tau$  and clients  $\mathcal{N}$ 
   using derandomization function Alg. 4.
19:   Remove all  $i$  from  $\mathcal{N}$ , if  $i \in \mathcal{N}$  is a qualified client, given coding vector  $\mathbf{a}_\tau$ .
20: end for

```

compare, over two sets of instances, the ratio between the benefit we get when we leverage side information and the benefit we get when we do not.

- *Ratio of $\frac{n}{t}$.* Assume that for each pair of clients i_1 and i_2 , the request sets R_{i_1} and R_{i_2} do not overlap, i.e., $R_{i_1} \cap R_{i_2} = \emptyset$ for all $i_1 \neq i_2$. Assume that each client receives a maximum score of s if she can decode her most preferred message. In this case, the best uncoded t selections are to choose the t messages such that t clients receive the maximum score, achieving benefit ts . With t encoded transmissions each client can decode her most preferred message. Therefore, the ratio is $ns/ts = n/t$.

- *Ratio of $\frac{2n}{t(k+1)}$.* Consider an instance with m messages and $n = m$ clients. All clients have a request set of the same size, i.e., $|R_i| = k < t, \forall i$. The clients are partitioned in groups: for any two clients i_1 and i_2 in different groups, their request sets R_{i_1} and R_{i_2} do not overlap, i.e., $R_{i_1} \cap R_{i_2} = \emptyset$; for any two clients i_1 and i_2 in the same group, their request sets R_{i_1} and R_{i_2} are the same, i.e., $R_{i_1} = R_{i_2}$. In each group, the number of clients equals to the cardinality of the request set k , and thus we have k clients requiring k messages. We assign the associated ranking submatrix of each group to be a Latin square, i.e., each required message is ranked differently by these k clients, from 1 to k . Hence, the ranking submatrix has no same elements in the same row or in the same column. Therefore, for the uncoded t selections, this instance will give a total score of $\frac{k(k+1)t}{2}$. For the coded t selections, this instance will give a total score

of nk , when we use MDS coding scheme to send all the missing messages. Hence, the ratio is $\frac{2n}{t(k+1)}$.

V. ARBITRARY SIZE SIDE INFORMATION (P3)

We are here given as input the score $s_i(j) = w_{ij}$ that client i has for message j , and make no assumptions on the size of the side information set. This is the most general case that admits P1 and P2 as special cases. We solve this problem using a mapping to the Maximum Weighted Independent Set (MWIS) problem¹.

Mapping to the MWIS problem: Assume that the server uses a binary coding vector $a = (a_1, \dots, a_m)$ to make a transmission $x = a_1b_1 + \dots + a_mb_m = b_{j_1} + b_{j_2} + \dots + b_{j_l}$, where the indices j_1, j_2, \dots, j_l correspond to the nonzero coding coefficients.

A client i can decode the message b_{j_1} from x if and only if this is the only message appearing in x that she does not have; that is, b_{j_1} belongs in her request set ($j_1 \in R_i$) and she has already as side information the rest of the messages appearing in x ($j_2, \dots, j_l \in S_i$). Consider now the $|R_i|$ positions in the coding vector a that correspond to the $|R_i|$ messages client i does not have. There are $|R_i|$ possible choices of coding coefficients for these positions, so that client i can decode one of these message: making exactly one of these coefficients one, and the remaining $|R_i| - 1$ zero. If we were to depict these coefficients sequentially, the choices are $(1, 0, 0, \dots, 0)_i, (0, 1, 0, \dots, 0)_i, \dots, (0, \dots, 0, 1)_i$, where we used the subscript i to express that these correspond to the messages in R_i . Client i can decode the first message in R_i under the first assignment, the second message under the second assignment, etc. We call these $|R_i|$ assignments the assignments for client i .

We map each of the $|R_i|$ assignments for client i , for all $i \in [n]$, to a vertex in the MWIS instance; thus in total we create $\sum_{i \in [n]} |R_i|$ vertices. We assign weight w_{ij} to the vertex corresponding to the assignment that enables client i to decode message j . We connect two vertices with an edge if the corresponding assignments cause conflict with each other: that is, there exists at least one common message to which one vertex assigns coefficient 0 and the other coefficient 1. Vertices corresponding to assignments of the same client i are pairwise connected, forming a clique, since these assignments are mutually exclusive. Vertices corresponding to assignments of different clients may be connected or not. For example, if

¹The MWIS problem is the weighted version of the maximum independent set problem. It aims to find a set of vertices that have the maximum weighted sum in a given graph. For details of the problem, see, for example, [19].

$R_1 = \{1, 2\}$ and $R_2 = \{2, 3, 4\}$, there are 5 assignments corresponding to clients 1 and 2, denoted as $(1, 0)_1, (0, 1)_1, (1, 0, 0)_2, (0, 1, 0)_2, (0, 0, 1)_2$. The vertex $(1, 0)_1$ is connected to $(0, 1)_1$ and $(1, 0, 0)_2$, where the latter is because $(1, 0)_1$ assigns a coding coefficient 0 to message 2 and $(1, 0, 0)_2$ assigns a coding coefficient 1 to message 2, resulting in a conflict.

Given that each vertex of this graph specifies part of a coding vector, an independent set specifies (perhaps in part) a feasible coding vector that enables all clients with a vertex in this independent set to decode a message. Thus, finding a MWIS enables to construct a coding vector that leads to the maximum benefit.

Algorithm for $t=1$: Given the MWIS connection, we can now translate any of the MWIS solvers to an algorithm for our problem when $t = 1$. As an example, the following theorem presents the score achievable by the MWIS polynomial time approximation algorithm in [19].

Theorem 6. *For problem P3, with $t = 1$ transmission, we can achieve a benefit of at least $\frac{W}{2(d_1-1)d_2+1}$ in polynomial time, where $W = \sum_{(i,j):j \in R_i} w_{ij}$ is the total weight of the instance, $d_1 = \max_{i \in [n]} \{|R_i|\} \leq m$, and $d_2 \leq n$ is the maximum number of request sets a message can belong to.*

Proof. The proof follows by observing that maximum degree of each vertex in the graph is at most $2(d_1 - 1)d_2$, and directly applying Theorem 3.4 in [19].

Indeed, consider a vertex v that enables client i to decode message $j \in R_i$. This vertex is connected to the remaining $|R_i| - 1$ vertices of the same client, which contributes to v degree at most $d_1 - 1$. Now consider another client $i' \neq i$. If $j \in R_{i'}$, then only one of the assignments for client i' does not have a conflict with v , the one that enables client i' to decode j . Thus counting each $i' \neq i$ we may have additional degree of at most $(d_2 - 1)(d_1 - 1)$. Finally, consider $j' \in R_i \cap R_{i'}$ for some $j' \neq j$. In this case, the vertex v' that enables client i' to decode message j' will be connected to v , since v' needs the coefficient of message j' to be 1 and v requires the coefficient of message j' to be 0. This last case contributes additional degree of at most $(d_1 - 1)d_2$. \square

Algorithm 3: This algorithm applies for general t and operates in t iterations, in each iteration simply solving one instance of a MWIS problem. We presented the algorithm in Alg. 3. In the first iteration, we solve the MWIS described earlier to select the transmission the server makes. Next, we update the problem instance: (i) we add decoded messages into side-information

Algorithm 3 Greedy Coding Algorithm to Solve P3.

```

1: Input: number of messages  $m$ , number of clients  $n$ , request sets  $R_i, \forall i \in [n]$ , weights  $w_{i,j}, \forall i \in [n], j \in R_i$ , number of encoded messages  $t$ .
2: Output:  $t$  encoded messages  $\{x_1, x_2, \dots, x_t\}$ .
3: Initialization: problem instance  $\mathcal{I} = (m, n, \{R_i\}_{i \in [n]}, \{w_{ij}\}_{i \in [n], j \in R_i})$ , received score  $s(i) = 0, \forall i \in [n]$ .
4: for  $\tau = 1 : t$  do
5:   Map the instance  $\mathcal{I}$  into an MWIS instance  $\mathcal{J}$ .
6:   Solve the MWIS problem  $\mathcal{J}$  and get output  $\{j_1, j_2, \dots, j_t\}$ .
7:   Set the  $\tau$ -th encoded message to be  $x_\tau = b_{j_1} + b_{j_2} + \dots + b_{j_t}$ .
8:   Update the instance  $\mathcal{I}$ :
9:   if Client  $i$  ( $\forall i \in [n]$ ) can decode message  $j \in R_i$  then
10:    Move  $j$  from  $R_i$  to  $S_i$ .
11:    if  $w_{ij} > 0$  then
12:      Update score:  $s(i) = w_{ij}$ .
13:      Set  $w_{ij'} = 0$  for all  $j' \in R_i$  and  $w_{ij'} \leq w_{ij}$ .
14:      Set  $w_{ij'} = w_{ij'} - w_{ij}$  for all  $j' \in R_i$  and  $w_{ij'} > w_{ij}$ .
15:    end if
16:  end if
17: end for

```

sets, and (ii) if client i has decoded message j , we set $w_{ij'} = \max\{0, w_{ij'} - w_{ij}\}$ for all $j' \in R(i)$, to reflect the additional benefit that receiving message j' would bring to client i given that she has already received j . We proceed with the next iteration by solving the MWIS problem on the new instance. Observe that this scenario admits the index coding problem as a special case, and hence we can show that the P3 problem is hard to approximate within a ratio of $n^{1-\epsilon}$ for any $\epsilon > 0$ (see Appendix C).

VI. NUMERICAL EVALUATION

A. Over random instances

For Figs. 1-8 we uniformly at random generate instances with the parameters described in the captions, and present values averaged over all instances. In the following experiments, we normalize the benefit B to get B_0 (divided by the maximum benefit possible to have maximum value 1).

Trade-off between B and t : Figs. 1 and 2 show for P1 (Alg. 1) and P2 (Alg. 2) the trade-off between the number of transmissions t and the normalized benefit B_0 . We consistently observe that we can achieve a large percentage of the benefit with a small fraction of the transmissions we need to achieve the maximum benefit. For example, in Fig. 1, a 2% decrease in benefit can

achieve a 71% bandwidth savings, and in Fig. 2, a 20% decrease in benefit can achieve a 91% bandwidth savings.

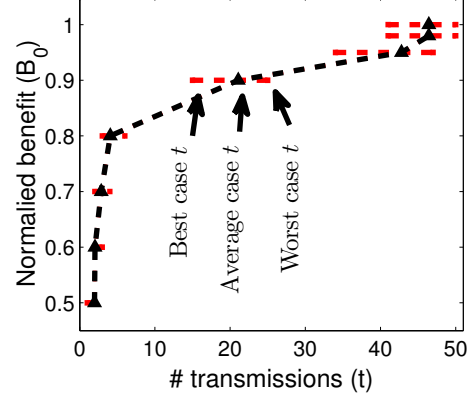
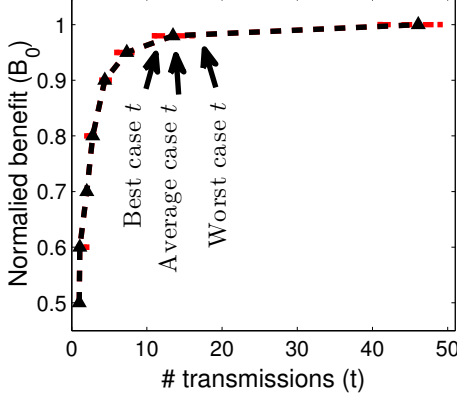


Fig. 1: Trade-off between bandwidth t and normalized benefit B_0 for Alg. 1 and P1 with normalized benefit B_0 for Alg. 2 and P2 with $m = 300$, $n = 50$ and Borda score model, $m = 300$, $n = 50$, $k = 60$ and Borda score averaged over 100 random instances. model, averaged over 100 random instances.

Benefit for small t : Figs. 3 and 4 highlight the normalized benefit B_0 we can achieve if the server is restricted to very few transmissions ($t = 1$ or $t = 4$) over some scenario. Observe that with $t = 4$ transmissions we can consistently achieve more than 85% of the benefit and with $t = 1$ more than 38% of the benefit.

TABLE I: Description of scenario

Scenario	Side information	Score model	Parameters	Algorithms
Scenario 1	No	Borda score	$m = 1000$, $n = 20$	Alg. 1
Scenario 2	No	Bimodal score with $G = 10$ and $F = 0.1$	$m = 1000$, $n = 20$	Alg. 1
Scenario 3	Yes, $k = 100$	Borda score	$m = 1000$, $n = 20$	Alg. 2
Scenario 4	Yes, $k = 100$	Bimodal score with $G = 10$ and $F = 0.1$	$m = 1000$, $n = 20$	Alg. 3

Benefit from coding: Figs. 5 and 6 compare, over two sets of parameters for P2, the performance of Alg. 1 (we run Alg. 1 by ignoring the side information and making uncoded transmissions) and Alg. 2. We find that leveraging the side information and coding enables Alg. 2 to double in some cases the benefit.

Benefit over random selection: Figs. 7 and 8 compare the performance of Alg. 1 with that of random selection over bimodal instances of P1. Random selection assumes that the server first identifies all messages that form first preference for at least one client, and then randomly

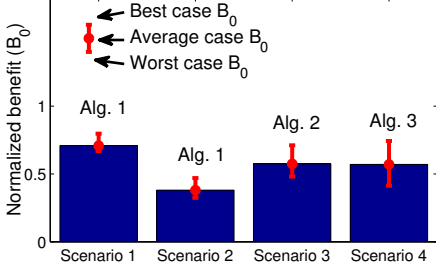


Fig. 3: Normalized benefit for $t = 1$. The scenarios are described in Table I.

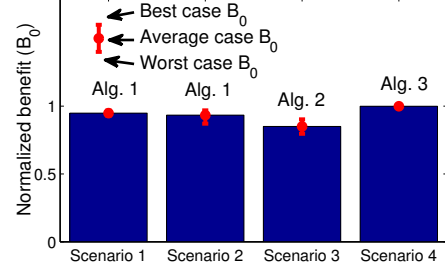


Fig. 4: Normalized benefit for $t = 4$. The scenarios are described in Table I.

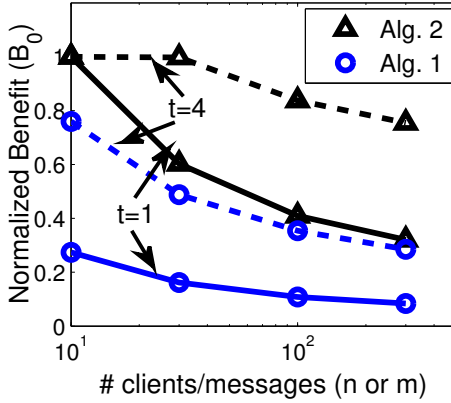


Fig. 5: We consider instances of P2 with $n = m$, $k = 0.1m$, and the Borda score model.

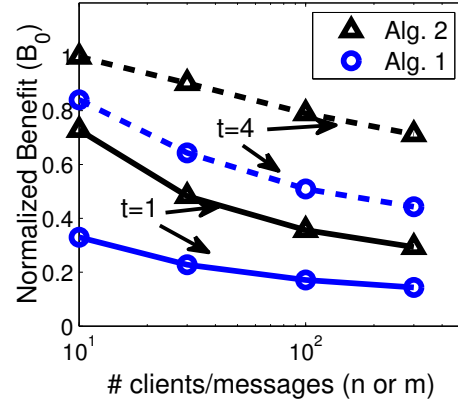


Fig. 6: We consider instances of P2 with $n = m$, $k = 0.2m$, and the Borda score model.

selects to transmit t of them. We find that Alg. 1 achieves 52% – 80% more benefit than random selection for $t = 4$, and 60% – 88% more for $t = 1$ as m changes from 10 to 1000; and achieves 30% – 71% more benefit than random selection for $t = 4$, and 31% – 106% more for $t = 1$ as G changes from 2 to 50.

B. Over data sets

We extract instances from the Yahoo! Search Marketing advertiser bidding data set, which was collected every 15 minutes over a year's period; data instances include the time stamp, the key phrase ID, the advertiser ID, and the bidding price [10]. We generate a problem instance as follows: During each hour, n users with n search query key phrases enter the system (these are the clients). The advertiser bids to place an ad (the ads are the messages) to some of the key phrases using certain prices (we assume price zero for the rest of the key phrases). We interpret the price of a bid for a key phrase as the score of this ad (message) with respect to

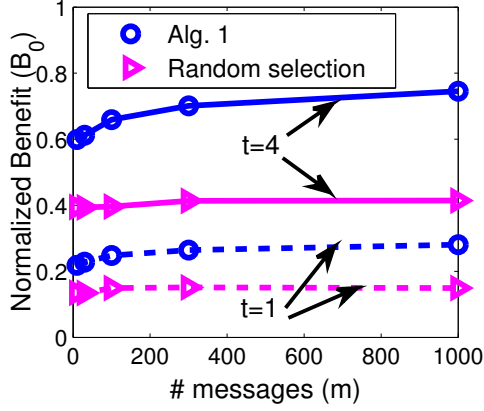


Fig. 7: We consider P1 with $n = 50$, and the bimodal score model with $G = 10$ and $F = 0.1$.

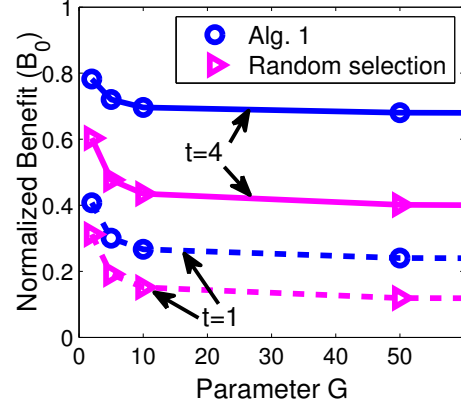


Fig. 8: We consider P1 with $m = 300$, $n = 50$, and the bimodal score model with $F = 0.1$, as G varies.

the key phrase (client). We assume that the messages that the advertiser does not bid for, form side information.

We compare Alg. 1 and 3 with the conventional Borda count method [9], the Spearfoot rule-based method [16], [18], and the Kemeny's method [17], [18] (we interpret the recommendations these algorithms make as the uncoded messages to transmit). The horizontal axis represents time (instances collected at sequential time slots). In Figs. 9 and 11 the vertical axis represents the actual benefit achieved at each time (current instance); in Figs. 10 and 12, the vertical axis represents the accumulated benefit (all previous instances). We find that all uncoded algorithms (including Alg. 1) perform similarly; this is because in the data set a few of the messages concentrated the highest rankings from all clients, and thus the score model used by the algorithm did not make a difference in the message choice. However, by leveraging the side information, Alg. 3 could accrue multiple times the benefit over time.

VII. CONCLUSION

In this paper, we have examined the recommendation benefit under bandwidth constraint in a index coding framework. We presented three problem examples to show that although the problems are in general NP hard, designing polynomial time approximation algorithms can still make a significant bandwidth savings by leveraging coding. We also conducted experiments over real data set to validate our arguments.

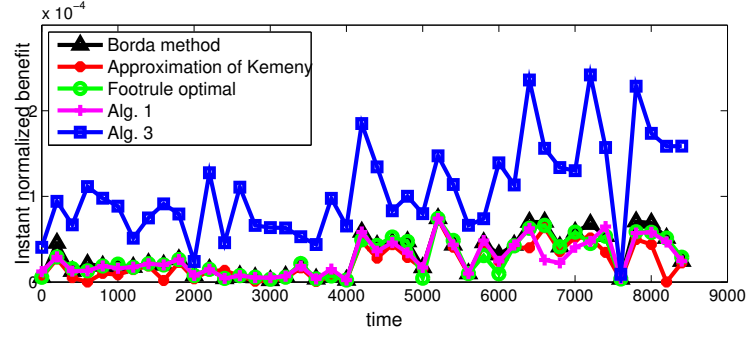


Fig. 9: Instant benefit for $t = 2$ as a function of time (current instance).

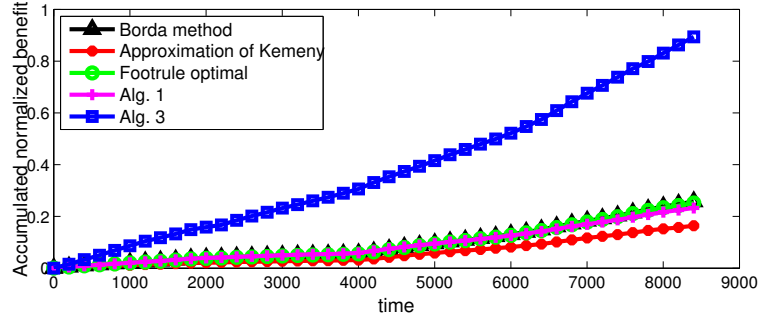


Fig. 10: Accumulated benefit for $t = 2$ over time (all previous instances).

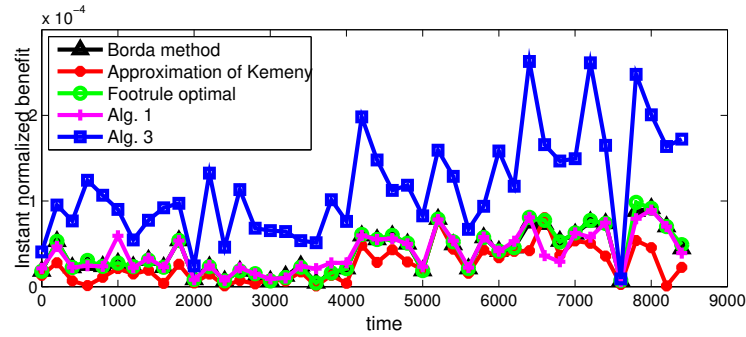


Fig. 11: Instant benefit for $t = 4$ as a function of time (current instance).

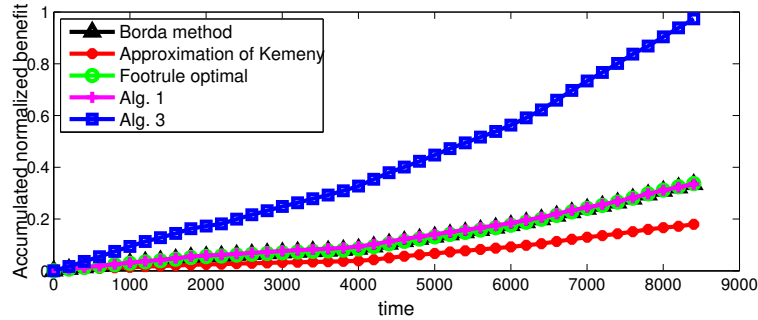


Fig. 12: Accumulated benefit for $t = 4$ as a function of time (all previous instances).

REFERENCES

- [1] P. Resnick and H. R. Varian, “Recommender systems,” *Communications of the ACM*, vol. 40, no. 3, pp. 56–58, 1997.
- [2] Y.-M. Li, C.-L. Chou, and L.-F. Lin, “A social recommender mechanism for location-based group commerce,” *Information Sciences*, vol. 274, pp. 125–142, 2014.
- [3] “Conviva viewer experience report,” www.conviva.com, 2013-2015.
- [4] Z. Bar-Yossef, Y. Birk, T. Jayram, and T. Kol, “Index coding with side information,” *IEEE Transactions on Information Theory*, vol. 57, no. 3, pp. 1479–1494, 2011.
- [5] Y. Birk and T. Kol, “Informed-source coding-on-demand (ISCOD) over broadcast channels,” in *17th Annual Joint Conference of the IEEE Computer and Communications Societies (INFOCOM '98)*, vol. 3, pp. 1257–1264, 1998.
- [6] A. Blasiak, R. Kleinberg, and E. Lubetzky, “Index coding via linear programming,” *arXiv preprint arXiv:1004.1379*, 2010.
- [7] S. El Rouayheb, A. Sprintson, and C. Georghiades, “On the index coding problem and its relation to network coding and matroid theory,” *IEEE Transactions on Information Theory*, vol. 56, pp. 3187–3195, 2010.
- [8] M. Effros, S. El Rouayheb, and M. Langberg, “An equivalence between network coding and index coding,” *IEEE Transactions on Information Theory*, vol. 61, no. 5, pp. 2478–2487, 2015.
- [9] J. C. de Borda, “Mémoire sur les élections au scrutin,” 1781.
- [10] “Yahoo! search marketing advertiser bidding data, version 1.0,” website, <https://webscope.sandbox.yahoo.com/catalog.php?datatype=a>. accessed: 2016-05-30.
- [11] E. Lubetzky and U. Stav, “Nonlinear index coding outperforming the linear optimum,” *IEEE Transactions on Information Theory*, vol. 55, no. 8, pp. 3544–3551, 2009.
- [12] I. Haviv and M. Langberg, “On linear index coding for random graphs,” in *Proceedings 2012 IEEE International Symposium on Information Theory (ISIT)*, pp. 2231–2235, IEEE, 2012.
- [13] A. Golovnev, O. Regev, and O. Weinstein, “The minrank of random graphs,” *arXiv preprint arXiv:1607.04842*, 2016.
- [14] L. Song and C. Fragouli, “A polynomial-time algorithm for pliable index coding,” in *Proceedings 2016 IEEE International Symposium on Information Theory (ISIT)*, 2016.
- [15] S. Brahma and C. Fragouli, “Pliable index coding,” *IEEE Transactions on Information Theory*, vol. 61, no. 11, pp. 6192–6203, 2015.
- [16] P. Diaconis and R. L. Graham, “Spearman’s footrule as a measure of disarray,” *Journal of the Royal Statistical Society. Series B (Methodological)*, pp. 262–268, 1977.
- [17] J. G. Kemeny, “Mathematics without numbers,” *Daedalus*, vol. 88, no. 4, pp. 577–591, 1959.
- [18] C. Dwork, R. Kumar, M. Naor, and D. Sivakumar, “Rank aggregation methods for the web,” in *Proceedings of the 10th International Conference on World Wide Web*, pp. 613–622, ACM, 2001.
- [19] S. Sakai, M. Togasaki, and K. Yamazaki, “A note on greedy algorithms for the maximum weighted independent set problem,” *Discrete Applied Mathematics*, vol. 126, no. 2, pp. 313–322, 2003.
- [20] M. Langberg and A. Sprintson, “On the hardness of approximating the network coding capacity,” *IEEE Transactions on Information Theory*, vol. 57, no. 2, pp. 1008–1014, 2011.
- [21] J. Håstad, “Clique is hard to approximate within $n^{1-\epsilon}$,” in *Proceedings 37th Annual Symposium on Foundations of Computer Science (FOCS '96)*, pp. 627–636, IEEE, 1996.

APPENDIX A
APPENDIX FOR SECTION III (P1)

A. Proof of Theorem 1

Consider a set cover problem instance, with a universe set $U = \{1, 2, \dots, n\}$ and a family of subsets of U , $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$. The union of elements of \mathcal{S} is U , i.e., $S_1 \cup S_2 \cup \dots \cup S_k = U$ and $S_j \subseteq U$ for all $j \in [k]$. The goal of the set cover problem is to find a subset of \mathcal{S} , $\mathcal{S}' \subseteq \mathcal{S}$, with minimum cardinality such that $\cup_{S \in \mathcal{S}'} S = U$.

We show that a set cover problem instance can be reduced to a P1 problem instance with m messages and n clients in polynomial time. The clients will correspond to elements in the universe U , and we will have $m = (n + 2)(nk - n + 1)$ messages, that correspond to the k subsets in the family \mathcal{S} plus some additional (dummy) messages for construction purposes. We will construct a corresponding $n \times m$ ranking matrix Π , and sequentially test whether a selection of $t = 1, 2, \dots, k$ columns can achieve a benefit greater than or equal to $n(m + 1 - k)$. Recall that in the ranking matrix the rows correspond to clients (elements of U), and the columns to messages (each one of the first k columns will represent a subset in the family \mathcal{S} and the remaining the dummy messages).

Let us denote by $\mathcal{S}[i]$ the family of subsets that contains element $i \in U$, i.e., $\mathcal{S}[i] = \{S \in \mathcal{S} : i \in S\}$. We denote the cardinality of $\mathcal{S}[i]$ by $d_i \leq k$ and the columns corresponding to subsets in $\mathcal{S}[i]$ by $j_{i1}, j_{i2}, \dots, j_{id_i}$. Let us define $g = nk - n + 1$ and $d = d_1 + d_2 + \dots + d_n$. Then $m = (n + 2)g$. The ranking matrix we would like to construct has the following form:

$$\Pi = [\Pi_0, \Pi_1, \Pi_2, \dots, \Pi_n, R], \quad (8)$$

where Π_0 is of size $n \times k$; Π_i ($1 \leq i \leq n$) is of size $n \times (g - d_i)$; and R is of size $n \times (2g - k + d)$. The construction process is as follows.

- Step 1: construct submatrix Π_0 . We assign rankings row by row to the matrix. For the i -th row, assign an arbitrary permutation of ranks $1, 2, \dots, d_i$ to d_i positions corresponding to $\mathcal{S}[i]$, and assign an arbitrary permutation of ranks $g + 1, g + 2, \dots, g + k - d_i$ to $(k - d_i)$ positions corresponding to $\mathcal{S}^C[i] = \mathcal{S} \setminus \mathcal{S}[i]$.

- Step 2: construct submatrix Π_i , $1 \leq i \leq n$, of size $n \times (g - d_i)$. First, assign an arbitrary permutation of the ranks $d_i + 1, d_i + 2, \dots, g$ as the row vector of row i . For the other rows $l \neq i$,

assign an arbitrary permutation of the ranks $(i+1)g+1, (i+1)g+2, \dots, (i+1)g+(g-d_i)$ as the row vector of row l .

• Step 3: construct submatrix R . For each row i of submatrix R , assign an arbitrary permutation of the remaining $(2g-k+d)$ ranks $g+k-d_i+1, g+k-d_i+2, \dots, 2g, 2g+g-d_1+1, \dots, 3g, 3g+g-d_2+1, \dots, 4g, \dots, (n+1)g+g-d_n+1, \dots, (n+2)g, (i+1)g+1, (i+1)g+2, \dots, (i+1)g+(g-d_i)$ as a row vector of row i .

Example 1: We illustrate the construction of matrix Π using a simple example. Let us consider a set cover problem represented by the following adjacency matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

where each element of the universe $U = \{1, 2, 3\}$ is represented by a row and each subset in the subset family $\mathcal{S} = \{S_1 = \{1, 3\}, S_2 = \{1, 2\}, S_3 = \{2, 3\}, S_4 = \{1\}\}$ is represented by a column. In this case, we have $n = 3$, $k = 4$, $g = nk - n + 1 = 10$, $m = (n+2)g = 50$. Then we can construct our ranking matrix in the following form:

$$\Pi = \left[\begin{array}{cccc|cccc|cccc|cccc|cccc} 1 & 2 & 11 & 3 & 4 & 5 & \dots & 10 & 31 & 32 & \dots & 38 & 41 & 42 & \dots & 48 & 12 & \dots \\ 11 & 1 & 2 & 12 & 21 & 22 & \dots & 27 & 3 & 4 & \dots & 10 & 41 & 42 & \dots & 48 & 13 & \dots \\ 1 & 11 & 2 & 12 & 21 & 22 & \dots & 27 & 31 & 32 & \dots & 38 & 3 & 4 & \dots & 10 & 13 & \dots \end{array} \right].$$

$\Pi_0 \qquad \qquad \Pi_1 \qquad \qquad \Pi_2 \qquad \qquad \Pi_3 \qquad \qquad R$

Next, we prove that finding a cover of size t can be achieved by finding t columns that achieve a certain benefit; hence we can sequentially test whether t is the minimum set cover size, from which the theorem follows. We map a selection of t sets for the cover problem to the selection of the corresponding messages (columns of Π_0) to transmit; and reversely, when selecting messages to transmit, we consider as part of the cover the sets corresponding to messages/columns of Π_0 (we ignore dummy messages). Thus we say that selection of t columns to transmit “covers” a client, if the corresponding set in the set cover problem includes this client.

Lemma 1. *The selection of t subsets in \mathcal{S} can cover the universe U , if and only if we can find*

t columns in matrix Π that achieve benefit at least $n(m + 1 - k)$.

Proof. • **Necessity.** If a selection of t subsets in \mathcal{S} can cover the universe U , this selection can achieve at least a minimum rank of k for any client, resulting in a benefit at least $n(m + 1 - k)$ according to our ranking assignment in construction step 1.

• **Sufficiency.** Suppose there exists an optimal selection \mathcal{T} of size t that does not cover some client $i \in [n]$ and achieves a benefit $B_{\mathcal{T}} \geq n(m + 1 - k)$.

We observe that the selection \mathcal{T} does not contain any column in Π_i .

Indeed, if the selection \mathcal{T} does not cover client i , but contains a column j in Π_i , then we can simply construct another selection by replacing column j with the column that contains the rank 1 choice of client i . Formally, we can construct $\mathcal{T}' = \mathcal{T} \cup \{\pi_i^{-1}(1)\} \setminus \{j\}$, where $\pi_i^{-1}(1)$ is the column that contains the rank 1 choice of client i . From construction, we can see that the ranks of message $\pi_i^{-1}(1)$ (column $\pi_i^{-1}(1)$) is better than j for all clients and at least client i can improve her score. So \mathcal{T}' is a better selection than \mathcal{T} , resulting in a contradiction of \mathcal{T} being an optimal selection.

Thus it suffices to consider the case that \mathcal{T} does not cover client i and does not contain any column in Π_i . From construction, we note that columns corresponding to $\mathcal{S}[i]$ and Π_i contain choices of ranks $1, 2, \dots, g$ of client i . As a result, the minimum rank that client i can achieve is $g + 1$, and the maximum score client i can achieve is $m - g$. Therefore, the benefit can be bounded by

$$B_{\mathcal{T}} \leq (n - 1)m + (m - g) = n(m + 1 - k) - 1 < n(m + 1 - k), \quad (9)$$

which results in a contradiction with $B_{\mathcal{T}} \geq n(m + 1 - k)$.

We finally argue that a selection \mathcal{T} of t columns of the ranking matrix Π that can cover all $i \in U$, does not contain columns outside Π_0 . This follows from construction, since a column j_1 in Π_i ($i = 1, 2, \dots, n$) is dominated by any column in Π_0 that can cover client i , such that the removal of j_1 will not affect the benefit. This means that $\mathcal{T} \setminus \{j_1\}$ can achieve the same benefit as \mathcal{T} . In this case, the testing would stop at most at step $t - 1$. Similarly, a column j_2 in R is dominated by any column in Π_0 such that the removal of j_2 will not affect the benefit. \square

Example 2: Continuing with our previous example 1, if we choose t columns, we can see that only when t columns covering at least one column in $\{1, 2, 4\}$ (i.e., covering element 1), at least

one column in $\{2, 3\}$ (i.e., covering element 2), and at least one column in $\{1, 3\}$ (i.e., covering element 3) we can make the benefit no less than $n(m + 1 - k) = 141$.

B. Proof of Theorem 2

We use $B(\tau)$ and $\delta(\tau)$, $\tau = 1, 2, \dots, t$, to denote the benefit collected after τ steps and the increase of benefit in the τ -th step, respectively, using Alg. 1. That is:

$$\begin{aligned} B(1) &= \delta(1); \\ B(2) &= B(1) + \delta(2); \\ &\vdots \\ B_T &= B(t) = B(t-1) + \delta(t). \end{aligned} \tag{10}$$

We set $B(0) = 0$ and denote by B^* the optimal benefit and by \mathcal{T}^* an optimal selection of t messages (columns of the ranking matrix) to transmit that achieve B^* . For all τ , we can bound $\delta(\tau)$ using the following lemma.

Lemma 2. *In Alg. 1, we can bound the increase in benefit in each step by:*

$$\delta(\tau) \geq \frac{B^* - B(\tau-1)}{t}, \quad \tau = 1, 2, \dots, t. \tag{11}$$

Proof. To prove this lemma, we first observe the following. Consider two selections of messages (columns of the ranking matrix) \mathcal{T}_1 and \mathcal{T}_2 , with $\mathcal{T}_1 \subseteq \mathcal{T}_2$. If we add a new column j to these selections, we get that:

$$B_{\mathcal{T}_1 \cup \{j\}} - B_{\mathcal{T}_1} \geq B_{\mathcal{T}_2 \cup \{j\}} - B_{\mathcal{T}_2}. \tag{12}$$

Indeed, when we add message j (column j) to \mathcal{T}_1 , assume C is the set of clients whose score can be improved. The left hand side of eq. (12), denoted by δ_1 , is the increase in score due to this set of clients C and to a level determined by their ranking of message j . Because we have $\mathcal{T}_1 \subseteq \mathcal{T}_2$, when we add message j (column j) to \mathcal{T}_2 , clearly clients not in C cannot achieve a higher score because of j and clients in C cannot achieve a benefit improvement as large as δ_1 (since some of the clients in C may have already higher scores thanks to messages in $\mathcal{T}_2 \setminus \mathcal{T}_1$).

Going back to the proof of (11), observe that at the beginning of the τ -th step, the benefit difference from the optimal benefit is $B^* - B(\tau-1)$. From the pigeonhole principle, at least one of the t messages in the optimal set \mathcal{T}^* , let us say j , can improve the benefit at least by

$\frac{B^* - B(t-1)}{t}$, since a future selection of j can only offer improvements less than or equal to what it can offer in the current stage from (12). \square

Using Lemma 2 we can prove the bound of our theorem.

$$\begin{aligned}
B(t) &= B(t-1) + \delta(t) \\
&\geq B(t-1) + \frac{B^* - B(t-1)}{t} \\
&= (1 - \frac{1}{t})B(t-1) + \frac{1}{t}B^* \\
&\geq (1 - \frac{1}{t})[(1 - \frac{1}{t})B(t-2) + \frac{1}{t}B^*] + \frac{1}{t}B^* \\
&= (1 - \frac{1}{t})^2 B(t-2) + \frac{1}{t}[1 + (1 - \frac{1}{t})]B^* \\
&\geq \dots \\
&\geq (1 - \frac{1}{t})^{t-1} \frac{B^*}{t} + \frac{1}{t}[1 + (1 - \frac{1}{t}) + (1 - \frac{1}{t})^2 + \dots + (1 - \frac{1}{t})^{t-2}]B^* \\
&= (1 - \frac{1}{t})^{t-1} \frac{B^*}{t} + [1 - (1 - \frac{1}{t})^{t-1}]B^* \\
&= [1 - (1 - \frac{1}{t})^t]B^* \\
&\geq [1 - \frac{1}{e}]B^*,
\end{aligned} \tag{13}$$

where the first four inequalities hold by repeatedly using Lemma 2; the second to the last equality holds according to the finite geometric series calculation; the last inequality holds according to the inequality $(1 - \frac{1}{t})^t \leq 1/e$.

C. Proof of Theorem 3

Lower bound: We prove the lower bound by showing that a simple random choice of t columns achieves expected benefit $\mu \triangleq \frac{tn(m+1)}{t+1}$ and thus, there exists a selection of t columns that achieves at least such benefit, $B \geq \mu$.

Consider an $n \times m$ ranking matrix Π . Assume we select uniformly at random t columns from all m columns, i.e., with equal probability select 1 from all $\binom{m}{t}$ possible selections. We will calculate the expected benefit from this random selection.

For a given selection, if we denote by X_1, X_2, \dots, X_n the score received by clients $1, 2, \dots, n$, then the benefit is $B = X_1 + X_2 + \dots + X_n$. Note that since we randomly select the t columns and the rows can have arbitrary assignments of ranking for each client, the expected scores are equal, namely, $\mathbb{E}X_1 = \mathbb{E}X_2 = \dots = \mathbb{E}X_n$. Thus it is sufficient to calculate $\mathbb{E}X_1$.

Denote by \mathcal{T} a realization of the t -selection of columns. We next just consider all $(m - t + 1)$ possibilities for the score X_1 that client 1 has achieved. We will use the notation $\pi_i^{-1}(j)$ to refer to the message ranked j by client i .

- Case 1: the message ranked 1 is in \mathcal{T} , i.e., $\pi_1^{-1}(1) \in \mathcal{T}$, and $X_1 = m$. The probability of $\{\pi_1^{-1}(1) \in \mathcal{T}\}$ is $p_1 = \frac{t}{m}$.
- Case 2: the message ranked 1 is in $[m] \setminus \mathcal{T}$ and the message ranked 2 is in \mathcal{T} , i.e., $\{\pi_1^{-1}(2) \in \mathcal{T} \wedge \pi_1^{-1}(1) \notin \mathcal{T}\}$. In this case, $X_1 = m - 1$ and occurs with probability $p_2 = \frac{t(m-t)}{m(m-1)}$, where $\frac{t}{m}$ is the probability of $\{\pi_1^{-1}(2) \in \mathcal{T}\}$ and $\frac{m-t}{m-1}$ is the probability of $\{\pi_1^{-1}(1) \notin \mathcal{T}\}$.

Continuing along these lines we get:

- Case j : the messages ranked $1, 2, \dots, j-1$ are in $[m] \setminus \mathcal{T}$ and the message ranked j is in \mathcal{T} , i.e., $\{\pi_1^{-1}(j) \in \mathcal{T} \wedge \pi_1^{-1}(1) \notin \mathcal{T} \wedge \dots \wedge \pi_1^{-1}(j-1) \notin \mathcal{T}\}$. In this case, $X_1 = m + 1 - j$, and occurs with probability $p_j = \frac{t(m-t)(m-t-1)\dots(m-t-j+2)}{m(m-1)(m-2)\dots(m-j+1)}$, where $\frac{t}{m}$ is the probability of $\{\pi_1^{-1}(j) \in \mathcal{T}\}$, and $\frac{(m-t)(m-t-1)\dots(m-t-j+2)}{(m-1)(m-2)\dots(m-j+1)}$ is the probability of $\{\pi_1^{-1}(1) \notin \mathcal{T} \wedge \dots \wedge \pi_1^{-1}(j-1) \notin \mathcal{T}\}$. Hence, we have

$$\begin{aligned} \mathbb{E}X_1 &= p_1 m + p_2(m-1) + \dots + p_j(m+1-j) + \dots + p_{m-t+1}t \\ &= \frac{t}{m!} \sum_{j=1}^{m-t+1} [P(m-t, j-1)(m-j+1)!] = \frac{t(m+1)}{t+1}, \end{aligned} \quad (14)$$

where the third equality holds from Lemma 3 that we provide later in this appendix.

Upper bound: We use a probabilistic method to construct a $n \times m$ preference matrix instance Π as follows. Draw a permutation of $[m]$ from all $m!$ possible permutations iid uniformly at random, and assign it to the i -th row of the ranking matrix Π , for all $i = 1, 2, \dots, n$.

Consider a fixed t -selection of columns, e.g., $\mathcal{T} = [t] = \{1, 2, \dots, t\}$. Assume X_1, X_2, \dots, X_n are the scores received by clients $1, 2, \dots, n$, then the benefit is $B = X_1 + X_2 + \dots + X_n$. Due to the iid uniform selection of each row of Π , we have that $\mathbb{E}X_1 = \mathbb{E}X_2 = \dots = \mathbb{E}X_n$. We calculate $\mathbb{E}X_1$, by listing all the possibilities for client 1's (first row of the ranking matrix) rankings and scores. Recall that we use $\pi_1^{-1}(j)$ to denote the column (message) that is ranked j by client 1, and that the score of receiving a ranking j message is $m + 1 - j$. We have the following $(m - t + 1)$ possibilities for X_1 .

- Case 1: the message ranked 1 is in $[t]$, i.e., $\{\pi_1^{-1}(1) \in [t]\}$. In this case $X_1 = m$, and the probability of this event equals $p_1 = \frac{t(m-1)!}{m!}$, where t is the number of ways we can have the message ranked 1 selected in $[t]$; $(m-1)!$ is the number of ways we can rank the remaining $(m-1)$ messages; and the total number of possible assignments is $m!$. We underline that each

assignment occurs with equal probability.

- Case 2: the message ranked 1 is in $[m] \setminus [t]$ and the message ranked 2 is in $[t]$, i.e., $\{\pi_1^{-1}(2) \in [t] \wedge \pi_1^{-1}(1) \notin [t]\}$. In this case, $X_1 = m - 1$, and the associated probability is $p_2 = \frac{tP(m-t,1)(m-2)!}{m!}$, where t is the number of ways we can have the message ranked 2 in $[t]$; $P(m-t, 1)$, the 1-permutation of $m-t$, is the number of ways to assign the message ranked 1 in the remaining $m-t$ positions $t+1, t+2, \dots, m$; $(m-2)!$ is the number of ways we can assign the remaining $(m-2)$ messages; and the total number of possible assignments is $m!$.

Continuing along these lines we get:

- Case j : the messages ranked $1, 2, \dots, j-1$ are in $[m] \setminus [t]$ and the message ranked j is in $[t]$, i.e., $\{\pi_1^{-1}(j) \in [t] \wedge \pi_1^{-1}(1) \notin [t] \wedge \dots \wedge \pi_1^{-1}(j-1) \notin [t]\}$. In this case, $X_1 = m + 1 - j$, and this event occurs with probability $p_j = \frac{tP(m-t, j-1)(m-j)!}{m!}$, where t is the number of ways we can have the message ranked j in $[t]$; $P(m-t, j-1)$, the j -permutations of $m-t$, is the number of ways we can assign the messages ranked $1, 2, \dots, j-1$ in the remaining $m-t$ positions $t+1, t+2, \dots, m$; and $(m-j)!$ is the number of choices for the remaining $(m-j)$ messages.

Therefore:

$$\begin{aligned} \mathbb{E}X_1 &= p_1m + p_2(m-1) + \dots + p_j(m+1-j) + \dots + p_{m-t+1}t \\ &= \frac{t}{m!} \sum_{j=1}^{m-t+1} [P(m-t, j-1)(m-j+1)!] = \frac{t(m+1)}{t+1}, \end{aligned} \quad (15)$$

where the third equality holds from Lemma 3.

From the Chernoff bound, we can bound the probability that the benefit is above $B_{UPPER} = (1 + \delta)\mu$ (where $\delta = \sqrt{\frac{6t \log(m)}{n}}$):

$$\Pr\{B \geq (1 + \delta)\mu\} = \Pr\left\{\frac{B}{m} \geq \frac{1}{m}(1 + \delta)\mu\right\} \leq e^{-\frac{\mu\delta^2}{3m}} = e^{-\frac{2t^2(m+1)\log(m)}{m(t+1)}} \triangleq \epsilon, \quad (16)$$

where the first equality just normalizes the random variable B , i.e., $\frac{B}{m} = \frac{X_1}{m} + \frac{X_2}{m} + \dots + \frac{X_n}{m}$, such that $\frac{X_i}{m}$ is between 0 and 1. Note that $\mathbb{E}\frac{B}{m} = \frac{\mu}{m}$.

The inequality $\Pr\{B \geq (1 + \delta)\mu\} \leq \epsilon$ implies that for the fixed t -selection of columns, $[t]$, there are at most an ϵ fraction of instances when selecting the matrix Π that can achieve a benefit no less than $(1 + \delta)\mu$. Due to the uniform at random selection of Π , given any fixed t -selection, it is also the case that at most ϵ fraction of instances can achieve this benefit $(1 + \delta)\mu$. There are in total $\binom{m}{t}$ possible selections of columns. The fraction of instances of matrices Π that can

achieve a benefit no less than $(1 + \delta)\mu$ given any of the $\binom{m}{t}$ t -selections is at most

$$\binom{m}{t}\epsilon < m^t e^{-\frac{2t^2(m+1)\log(m)}{m(t+1)}} = e^{(t\log(m) - \frac{2t^2(m+1)\log(m)}{m(t+1)})} = e^{(-\frac{t\log(m)[(t-1)m+2t]}{m(t+1)})} < 1, \quad (17)$$

which indicates that there must exist instances, such that, for any t -selection, the average score cannot be more than B_{UPPER} . This concludes the proof of the theorem.

Lemma 3.

$$\frac{1}{m!} \sum_{j=1}^{m-t+1} [P(m-t, j-1)(m-j+1)!] = \frac{m+1}{t+1}, \text{ for any } t \leq m. \quad (18)$$

Proof. Change the variable of this equation by setting $k = m-t$. Denote by $H(k)$ the expression on the left hand side of eq. (18), i.e.,

$$H(k) = \frac{1}{m!} \sum_{j=1}^{k+1} [P(k, j-1)(m-j+1)!]. \quad (19)$$

To show that $H(k) = \frac{m+1}{m-k+1}$ for any $m \geq k$, we use a mathematical induction method for k and consider m as a parameter.

For $k = 0$, $\frac{1}{m!}P(0, 0)m! = 1$ and for $k = 1$, $\frac{1}{m!}(P(1, 0)m! + P(1, 1)(m-1)!) = 1 + \frac{1}{m}$, eq. (18) holds. Assume eq. (18) holds for $k \leq m-1$. Now, for $k+1$, we have

$$\begin{aligned} H(k+1) &= \frac{1}{m!} \sum_{j=1}^{k+2} [P(k+1, j-1)(m-j+1)!] \\ &= \frac{1}{m!} \left[\frac{(k+1)!}{(k+1)!} m! + \sum_{j=2}^{k+2} \frac{(k+1)!}{(k+1-j+1)!} (m-j+1)! \right] \\ &= 1 + \frac{k+1}{m(m-1)!} \sum_{l=1}^{k+1} \frac{k!}{(k+1-l)!} (m-1-l+1)! \\ &= 1 + \frac{(k+1)}{m} \cdot \frac{m-1+1}{m-1-k+1} \\ &= \frac{m+1}{m-k}, \end{aligned} \quad (20)$$

where the third equality holds due to a change of variable $l = j-1$ and the fourth equality holds due to the induction hypothesis on k with parameter $m-1$. Therefore, the equation (18) holds. \square

D. Proof of Theorem 4

We consider a family of instances \mathcal{I} , each with m messages and n clients. The ranking matrix Π is generated as follows: for each row, uniformly and independently draw a permutation from

all $m!$ permutations of $[m]$ and assign it to the row.

We first show that $\Delta(2, n, 1) = \frac{1}{\sqrt{2\pi n}}$ for even n and $\Delta(2, n, 1) = \frac{1}{\sqrt{2\pi(n-1)}}$ for odd $n \geq 3$.

Consider a $n \times 2$ ranking matrix instance Π in the family \mathcal{I} . Define n_1 and n_2 to be the numbers of 1s and 2s in the first column. Obviously, we have $n_1 + n_2 = n$ and n_1 and n_2 are the numbers of 2s and 1s in the second column. Our strategy is to select column 1 if $n_1 \geq n_2$ and to select column 2, otherwise. For even n , the expected benefit with respect to Π is

$$\mathbb{E}B = \sum_{n_1=0}^{n/2} \frac{1}{2^n} \binom{n}{n_1} [2(n - n_1) + n_1] + \sum_{n_1=n/2+1}^n \frac{1}{2^n} \binom{n}{n - n_1} [(n - n_1) + 2n_1], \quad (21)$$

where the first term corresponds to the selection of column 2 and the second term corresponds to the selection of column 1. Here, in the first term, $\binom{n}{n_1}$ is the number of choices for n_1 2s in the second column, resulting in a probability of $\frac{1}{2^n} \binom{n}{n_1}$ that the benefit is $2(n - n_1) + n_1$. The interpretation is similar for the second term. Similarly, for odd n , the expectation of benefit with respect to Π is

$$\mathbb{E}B = \sum_{n_1=0}^{\frac{n-1}{2}} \frac{1}{2^n} \binom{n}{n_1} [2(n - n_1) + n_1] + \sum_{n_1=\frac{n+1}{2}}^n \frac{1}{2^n} \binom{n}{n - n_1} [(n - n_1) + 2n_1]. \quad (22)$$

By simplifying this expression (see Lemma 4), we get that:

$$\mathbb{E}B = \begin{cases} \frac{3n}{2} + \frac{n}{\sqrt{2\pi n}}, & n \text{ even}, \\ \frac{3n}{2} + \frac{n}{\sqrt{2\pi(n-1)}}, & n \geq 3, \text{ odd}. \end{cases} \quad (23)$$

We next consider the general term $\Delta(m, n, t)$. The strategy we use here is as follows. We randomly select t columns. If this selection can achieve a benefit no less than $\mu = \frac{tn(m+1)}{t+1}$, we keep these columns as our selection. If not, we discard these columns, and select columns with a benefit at least μ . This is always possible according to Theorem 3.

Next, we look at when the benefit we actually achieve is greater than $\mu + \frac{n}{\sqrt{2\pi n}}\sigma(m, t)$.

For a fixed selection of t columns, if X_i is the score of client i , then the benefit X can be represented as $X = \sum_{i=1}^n X_i$. According to the central limit theorem, the distribution of $Y = \frac{X - \mu}{\sqrt{n\sigma(m, t)}}$ is approximately the standard normal distribution $\mathcal{N}(0, 1)$. Given our selection algorithm for A, the benefit is lower bounded by:

$$B \geq \begin{cases} \mu, & X \leq \mu, \\ X, & X > \mu. \end{cases} \quad (24)$$

Hence, the expected benefit can be lower bounded by

$$\mathbb{E}B \geq \mu + \Pr\{X > \mu\}\mathbb{E}[X - \mu|X > \mu]. \quad (25)$$

The second term $\Pr\{X > \mu\}\mathbb{E}[X - \mu|X > \mu]$ can be approximately calculated as:

$$\Pr\{X > \mu\}\mathbb{E}[X - \mu|X > \mu] \approx \int_0^\infty \sqrt{n}\sigma(m, t)y\phi(y)dy = \frac{n}{\sqrt{2\pi n}}\sigma(m, t), \quad (26)$$

where $\phi(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$ is the probability density function of the standard normal distribution.

The calculation of $\sigma(m, t)$ is in Lemma 5. From this the theorem follows.

In addition, we can strictly lower bound the average performance by using the Berry-Esseen theorem.

Corollary 1. *The expectation of benefit can be strictly lower bounded by $\mu + \frac{n}{\sqrt{2\pi n}}\sigma(m, t)(1 - e^{-\frac{m^2}{2}}) - \frac{m^3}{2\sigma^2(m, t)}$.*

Proof. The Berry-Esseen theorem states that the difference of distribution of $Y = \frac{(X - \mu)}{\sqrt{n}\sigma(m, t)}$ and the normal distribution can be bounded by

$$|F_Y(y) - \Phi(y)| \leq \frac{\rho}{2\sigma^3(m, t)\sqrt{n}}, \quad (27)$$

for all n and y , where $F_Y(y)$ and $\Phi(y)$ are the cumulative distribution functions of Y and the normal distribution, and $\rho = \mathbb{E}|X/n - \mathbb{E}X/n|^3 < m^3$. We also know that $|X - \mu| \leq mn$. Hence, the term $\Pr\{X > \mu\}\mathbb{E}[X - \mu|X > \mu]$ can be bounded by

$$\begin{aligned} \Pr\{X > \mu\}\mathbb{E}[X - \mu|X > \mu] &\geq \int_0^m \sqrt{n}\sigma(m, t)y\phi(y)dy - \sqrt{n}\sigma(m, t)\frac{\rho}{2\sigma^3(m, t)\sqrt{n}} \\ &= \frac{n}{\sqrt{2\pi n}}\sigma(m, t)(1 - e^{-\frac{m^2}{2}}) - \frac{m^3}{2\sigma^2(m, t)}. \end{aligned} \quad (28)$$

This proves the corollary. \square

Lemma 4.

$$\mathbb{E}B = \begin{cases} \frac{3n}{2} + \frac{n}{\sqrt{2\pi n}}, & n \text{ even}, \\ \frac{3n}{2} + \frac{n}{\sqrt{2\pi(n-1)}}, & n \geq 3, \text{ odd}. \end{cases} \quad (29)$$

Proof. When n is even, we have

$$\begin{aligned}
\mathbb{E}[B] &= \frac{1}{2^n} \left[\sum_{n_1=0}^{n/2} \binom{n}{n_1} (2(n - n_1) + n_1) + \sum_{n_2=0}^{n/2} \binom{n}{n_2} (2(n - n_2) + n_2) - \binom{n}{n/2} \left(\frac{3n}{2}\right) \right] \\
&= \frac{2}{2^n} \left[\sum_{n_1=0}^{n/2} \binom{n}{n_1} (2n - n_1) - \binom{n}{n/2} \left(\frac{3n}{4}\right) \right] \\
&= \frac{2}{2^n} \left[n(2^n + \binom{n}{n/2}) - n2^{(n-2)} - \binom{n}{n/2} \left(\frac{3n}{4}\right) \right] \\
&\approx \frac{3n}{2} + \frac{n}{\sqrt{2\pi n}},
\end{aligned} \tag{30}$$

where the last approximation is due to the Stirling's approximation and the third equality holds because of the following two equations:

$$2^n = \sum_{n_1=0}^{n/2} \binom{n}{n_1} + \sum_{n_1=n/2+1}^n \binom{n}{n_1} = 2 \sum_{n_1=0}^{n/2} \binom{n}{n_1} - \binom{n}{n/2}, \tag{31}$$

and

$$\sum_{n_1=0}^{n/2} \binom{n}{n_1} n_1 = n \sum_{n_1=1}^{n/2} \frac{(n-1)!}{(n_1-1)!(n-n_1)!} = n \sum_{k=0}^{n/2-1} \frac{(n-1)!}{k!(n-1-k)!} = n2^{(n-2)}. \tag{32}$$

When $n \geq 3$ is odd, we have

$$\begin{aligned}
\mathbb{E}[B] &= \frac{1}{2^{(n-1)}} \left[\sum_{n_1=0}^{(n-1)/2} \binom{n}{n_1} (2n - n_1) \right] \\
&= \frac{1}{2^{(n-1)}} \left[n2^n - \left(n2^{(n-2)} - \frac{n}{2} \binom{n-1}{(n-1)/2} \right) \right] \\
&\approx \frac{3n}{2} + \frac{n}{\sqrt{2\pi(n-1)}},
\end{aligned} \tag{33}$$

where the last approximation is due to the Stirling's approximation and the second equality holds because of the following two equations:

$$2^n = \sum_{n_1=0}^{(n-1)/2} \binom{n}{n_1} + \sum_{n_1=(n+1)/2}^n \binom{n}{n_1} = 2 \sum_{n_1=0}^{(n-1)/2} \binom{n}{n_1}, \tag{34}$$

and

$$\sum_{n_1=0}^{(n-1)/2} \binom{n}{n_1} n_1 = n \sum_{n_1=1}^{(n-1)/2} \frac{(n-1)!}{(n_1-1)!(n-n_1)!} = n \sum_{k=0}^{(n-3)/2} \binom{n-1}{k} = n \frac{2^{(n-1)} - \binom{n-1}{(n-1)/2}}{2}. \tag{35}$$

□

Lemma 5.

$$\sigma(m, t) = \sqrt{\frac{(m+1)(m-t)t}{(t+1)^2(t+2)}} \tag{36}$$

Proof. We already know that the expected value of the score X_1 is $\frac{t(m+1)}{t+1}$, and the distribution

of X_1 is as follows:

$$\Pr\{X_1 = m + 1 - j\} = \frac{tP(m-t, j-1)(m-j)!}{m!}, \quad j = 1, 2, \dots, m + 1 - t. \quad (37)$$

Next, we prove the following result using induction:

$$\mathbb{E}[X_1^2] = \sum_{j=1}^{m+1-t} \Pr\{X = m + 1 - j\}(m + 1 - j)^2 = \frac{t^2(m+1)}{t+1} + \frac{t(m-t)(m+1)}{t+2}, \quad (38)$$

or

$$\frac{1}{m!} \sum_{j=1}^{k+1} [P(k, j-1)(m-j+1)!(m-j+1)] = \frac{t(m+1)}{t+1} + \frac{(m-t)(m+1)}{t+2}. \quad (39)$$

We change the variable of this equation by setting $k = m - t$. We denote by $L(k)$ the following expression:

$$L(k) = \frac{1}{m!} \sum_{j=1}^{k+1} [P(k, j-1)(m-j+1)!(m-j+1)]. \quad (40)$$

When $k = 0$, the initial condition holds, i.e., $L(0) = m = \frac{m(m+1)}{m+1} + \frac{0(m+1)}{m-k+2}$. Assume that $L(k) = \frac{(m-k)(m+1)}{m-k+1} + \frac{k(m+1)}{t+2}$ holds for all $m > k$. Then, for $k + 1$, we have

$$\begin{aligned} L(k+1) &= \frac{1}{m!} \sum_{j=1}^{k+2} [P(k+1, j-1)(m-j+1)!(m-j+1)] \\ &= \frac{1}{m!} m!m + \frac{1}{m!} \sum_{j=2}^{k+2} \left[\frac{(k+1)!}{(k+1-j+1)!} (m-j+1)!(m-j+1) \right] \\ &= m + \frac{k+1}{m} \left[\frac{1}{(m-1)!} \sum_{l=1}^{k+1} \left[\frac{k!}{(k+1-l)!} (m-1-l+1)!(m-1-l+1) \right] \right] \\ &= m + \frac{k+1}{m} \left[\frac{(m-1-k)m}{m-k} + \frac{km}{m-k+1} \right] \\ &= \frac{(m-k-1)(m+1)}{m-k} + \frac{(k+1)(m+1)}{m-k+1}, \end{aligned} \quad (41)$$

where the third equality holds due to a change of variable $l = j - 1$ and the fourth equality holds due to the induction hypothesis on k with parameter $m - 1$. Therefore eq. (38) is proved.

Furthermore, we can calculate $\sigma^2(m, t) = \mathbb{E}[X_1^2] - \mu^2 = \frac{(m+1)(m-t)t}{(t+1)^2(t+2)}$. \square

APPENDIX B

APPENDIX FOR SECTION IV (P2)

Derandomization Function for Claim 1: We here describe a polynomial-time deterministic algorithm to select a coding vector α^ξ . We refer to the clients that can decode a message they

have ranked less than or equal to ξ as the *qualified clients*. For a given coding vector \mathbf{a} , we denote the number of qualified clients by $Y[\mathbf{a}]$.

We sequentially assign a coding coefficient 0 or 1 to the m coding coefficients in the vector $\mathbf{a} = [a_1 \ a_2 \ \dots \ a_m]$ in m steps. At the beginning of the j -th step, the first $j - 1$ coefficients have been assigned some values $a_1 = \bar{a}_1, a_2 = \bar{a}_2, \dots, a_{j-1} = \bar{a}_{j-1}$.

We define $Y_{\bar{\mathbf{a}}_{[j-1],0}} = \mathbb{E}_{\mathbf{a}} Y[\mathbf{a} | a_1 = \bar{a}_1, a_2 = \bar{a}_2, \dots, a_{j-1} = \bar{a}_{j-1}, a_j = 0]$ to be the expected number of qualified clients, averaged over all coding vectors with the assigned values for the first $j - 1$ coding coefficients $\bar{\mathbf{a}}_{[j-1]}$ and a 0 for the j -th coding coefficient; $Y_{\bar{\mathbf{a}}_{[j-1],1}} = \mathbb{E}_{\mathbf{a}} Y[\mathbf{a} | a_1 = \bar{a}_1, a_2 = \bar{a}_2, \dots, a_{j-1} = \bar{a}_{j-1}, a_j = 1]$ to be the expected number of qualified clients, averaged over all coding vectors with the assigned values for the first $j - 1$ coding coefficients $\bar{\mathbf{a}}_{[j-1]}$ and a 1 for the j -th coding coefficient; and $Y_{\bar{\mathbf{a}}_{[j]}} = \mathbb{E}_{\mathbf{a}} Y[\mathbf{a} | a_1 = \bar{a}_1, a_2 = \bar{a}_2, \dots, a_j = \bar{a}_j]$ to be the expected number of qualified clients, averaged over all coding vectors with the assigned values $\bar{\mathbf{a}}_{[j]}$ for the first j coefficients. We also use $\mathbf{a}_{[0]}$ to refer the empty set.

The algorithm proceeds as follows: *For step $j = 1, 2, \dots, m$, assign the j -th coding coefficient \bar{a}_j to be 1 if $Y_{\bar{\mathbf{a}}_{[j-1],1}} \geq Y_{\bar{\mathbf{a}}_{[j-1],0}}$ and 0 otherwise.*

In the derandomization function flow diagram, the steps 10-14 essentially implement the calculation of the expected values we use for the decision making. We track the probability that a client i will be a qualified client, p_i^{j-1} , for each step $j - 1$. Then we choose the coding coefficient a_j by comparing the expected numbers of qualified clients if we choose a 0 and a 1 for a_j . We set a state parameter z_i^j to represent the number of coefficient assignment patterns for the remaining messages in R_i such that client i can remain qualified. For example, $z_i^j = 3$ if $|\{j' \in R_i | j' > j, \pi_i(j) \leq \xi, a_{j''} = 0, \forall j'' \in R_i \text{ and } j'' \leq j\}| = 3$.

Derandomization function performance: We here argue that the coding vector $\bar{\mathbf{a}}$ we identify enables at least $\frac{n\xi}{ke}$ clients to be qualified, i.e., decode a message in their request set they have ranked less than or equal to ξ . Let $Y_{\bar{\mathbf{a}}}$ be the qualified clients after the derandomized function, and let Y be the qualified clients after the randomized selection in Claim 1, where we iid at random assigned value 0 to each coding coefficient with probability p . Tracking the qualified clients, originally we have $Y = (1 - p)Y_{\emptyset,0} + pY_{\emptyset,1}$; then $Y_{\emptyset,0} \geq Y$ or $Y_{\emptyset,1} \geq Y$ holds; and hence we have $Y_{\bar{\mathbf{a}}_{[1]}} \geq Y$. For step j , we can see that $Y_{\bar{\mathbf{a}}_{[j]}} = (1 - p)Y_{\bar{\mathbf{a}}_{[j-1],0} + pY_{\bar{\mathbf{a}}_{[j-1],1}}$. Hence, at least one of the two following inequalities, $Y_{\bar{\mathbf{a}}_{[j-1],0}} \geq Y_{\bar{\mathbf{a}}_{[j]}}$ and $Y_{\bar{\mathbf{a}}_{[j-1],1}} \geq Y_{\bar{\mathbf{a}}_{[j]}}$, holds. Therefore, using the derandomization function, we have $Y_{\bar{\mathbf{a}}_{[j]}} \geq Y_{\bar{\mathbf{a}}_{[j-1]}}$. Hence, $Y_{\bar{\mathbf{a}}} \geq Y \geq \frac{n\xi}{ke}$ holds.

Algorithm 4 Derandomization Function.

1: **Input:** number of messages m , number of clients n , request sets $R_i, \forall i \in [n]$, ranking $\pi_i(j), \forall i \in [n], j \in R_i$, size of request set k , and ranking threshold ξ .
2: **Output:** coding vector $\mathbf{a} \in \{0, 1\}^m$.
3: **Initialization:** set the client set $\mathcal{N} = [n]$; set the qualification probability $p_i^0 = \frac{\xi}{k}(1 - \frac{1}{k})^{k-1}$, for all client $i \in [n]$; set the state $z_i^0 = \xi$ for each client $i \in [n]$.
4: **for** $j = 1 : m$ **do**
5: **for** all $i \in \mathcal{N}$ **do**
6: **if** $j \notin R_i$ **then**
7: $p_{i,0}^j = p_{i,1}^j = p_i^{j-1}$; $z_{i,0}^j = z_{i,1}^j = z_i^{j-1}$. //A client is not affected if not connected to message j .
8: **end if**
9: **if** $j \in R_i$ and $\pi_i(j) \leq \xi$ **then**
10: // This is the case that client i ranks j no more than ξ .
 // Update the probability that client i can be qualified if a_j is 0 or 1:

$$p_{i,0}^j = \begin{cases} 0, & \text{if } z_i^{j-1} = 1 \text{ and } a_{j'} = 0 \text{ for all } j' < j \text{ and } j' \in R_i, \\ \frac{p_i^{j-1}}{1-1/k}, & \text{if } z_i^{j-1} = 1 \text{ and } a_{j'} = 1 \text{ for some } j' < j \text{ and } j' \in R_i, \\ \frac{(1-1/z_i^{j-1})p_i^{j-1}}{1-1/k}, & \text{otherwise;} \end{cases} \quad (42)$$

$$p_{i,1}^j = \begin{cases} 0, & \text{if } a_{j'} = 1 \text{ for some } j' < j \text{ and } j' \in R_i, \\ \frac{k p_i^{j-1}}{z_i^{j-1}}, & \text{otherwise.} \end{cases} \quad (43)$$

11: // Update the state of client i if a_j is 0 or 1:

$$z_{i,0}^j = \begin{cases} 0, & \text{if } z_i^{j-1} = 1 \text{ and } a_{j'} = 0 \text{ for all } j' < j \text{ and } j' \in R_i, \\ 1, & \text{if } z_i^{j-1} = 1 \text{ and } a_{j'} = 1 \text{ for some } j' < j \text{ and } j' \in R_i, \\ z_i^{j-1} - 1, & \text{otherwise;} \end{cases} \quad (44)$$

$$z_{i,1}^j = \begin{cases} 0, & \text{if } a_{j'} = 1 \text{ for some } j' < j \text{ and } j' \in R_i, \\ 1, & \text{otherwise.} \end{cases} \quad (45)$$

12: **end if**
13: **if** $j \in R_i$ and $\pi_i(j) > \xi$ **then**
14: // This is the case that client i ranks j more than ξ .
 // Update the probability that client i can be qualified if a_j is 0 or 1:

$$p_{i,0}^j = \frac{p_i^{j-1}}{1-1/k}; \quad p_{i,1}^j = 0. \quad (46)$$

15: // Update the state of client i if a_j is 0 or 1:

$$z_{i,0}^j = z_i^j; \quad z_{i,1}^j = 0. \quad (47)$$

16: **end if**
17: **end for**
18: **if** $\sum_{i \in \mathcal{N}: j \in R_i} p_{i,1}^j \geq \sum_{i \in \mathcal{N}: j \in R_i} p_{i,0}^j$ **then**
19: Set $a_j = 1$, $p_i^j = p_{i,1}^j$, and $z_i^j = z_{i,1}^j$.
20: **else**
21: Set $a_j = 0$, $p_i^j = p_{i,0}^j$, and $z_i^j = z_{i,0}^j$.
22: **end if**
23: Remove i from \mathcal{N} , if $z_i^j = 0$ for all $i \in \mathcal{N}$.
24: **end for**

APPENDIX C

APPENDIX FOR SECTION V (P3)

We here discuss the hardness of approximation for P3. Note that P3 admits the index coding as a special case, where each client requires one message with score 1 and others with score

0. The hardness of approximating index coding capacity is shown through a reduction from the maximum independent set problem [20]. Here, we use a similar reduction to show the hardness of approximating B^* as follows.

Proposition 1. *The P3 problem is hard (unless $NP = ZPP$) to approximate within a ratio of $n^{1-\epsilon}$ for any $\epsilon > 0$.*

Proof. For this we simply use the result that the MIS is hard (unless $NP = ZPP$) to approximate within a ratio of $n^{1-\epsilon}$ for any $\epsilon > 0$ [21]. We map an MIS instance on a graph $\mathcal{G} = (V, E)$, into a P3 problem as follows.

- We create a P3 instance with $m = n = |V|$: we map each of the $|V|$ vertices in \mathcal{G} to a client in P3; and we also create $|V|$ messages.
- A client i has message i in her request set with score $w_{ii} = 1$.
- A client i has a message $j \neq i$ in her request set R_i with score $w_{ij} = 0$ if and only if there is an edge between vertex i and vertex j in \mathcal{G} . All the remaining messages are in the side information set of client i .

We next argue that the size of the maximum independent set in \mathcal{G} equals the maximum benefit that we can achieve over the constructed P3 instance if we are restricted to $t = 1$ transmission. Indeed, given an independent set \mathcal{S} in \mathcal{G} , we can construct a coding vector for P3 that enables each client i in \mathcal{S} to decode message i (we simply use coding coefficients 1 for all such $i \in \mathcal{S}$ and 0 for the remaining coding coefficients). Recall that with one transmission, a client can decode a message i in her request set if and only if the coding coefficient for this message is nonzero and the coding coefficients for all other messages in her request set are zero. This would achieve benefit $|\mathcal{S}|$ in P3.

We argue that this is the maximum benefit we could achieve in P3: indeed, if a larger benefit was possible in P3, more than $|\mathcal{S}|$ clients i would have been able to decode their corresponding message i . Note also that, a coding vector that achieves a maximum benefit B^* , enables B^* clients i to decode their corresponding message i , and thus directly determines an independent set of size $|\mathcal{S}| = B^*$ in \mathcal{G} . \square